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FIRE DISTRIBUTION IN LANCHESTER INERTIAL COMBAT, I: 'SQUARE-LAW--ETC(U)
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FIRE DISTRIBUTION IN LANCHESTER INERTIAL COMBAT, I:

"SQUARE-LAW" ATTRITION OF TARGET TYPES

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(3) The influence of command and control limitations on fire distribution tactics for a homogeneous force in combat against heterogeneous enemy forces is studied through a deterministic optimal control problem. Lanchester-type equations for a "square law" attrition process are used to model the combat. Command and control limitations are incorporated into the model through upper and lower bounds on the rate at which the distribution of fire can be changed. The structure of the optimal fire distribution policy is examined. → next page

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It is shown that such command and control limitations do not essentially alter the optimal fire distribution decision rules, although the shifting of fires is initiated earlier when command and control limitations exist than when an entire force can instantaneously shift their fires from one target type to another. Thus, when there is "inertia" to overcome in shifting fires, one begins to change the distribution of fire before target priorities change in anticipation of this coming change. The theory of state variable inequality constraints plays a major role in solving this problem. Of particular mathematical difficulty is the presence of a second order state variable inequality constraint in the problem.

1. Introduction.

In all the fire distribution problems that we have considered previously [14]-[19] an implicit assumption has always been that fire could be instantaneously shifted from one target type to another. To illustrate, let us recall a typical problem:

$$\underset{\phi(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\},$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2, \quad (1)$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

and with initial conditions

$$x_1(t=0) = x_1^o, \quad x_2(t=0) = x_2^o, \quad y(t=0) = y_0.$$

In this problem ϕ (the fraction of the Y-forces which fires at X_1) is the control (decision or policy) variable. The reader should note that although the control must satisfy the condition $0 \leq \phi \leq 1$, the rate of change of ϕ is unrestricted so that ϕ can instantaneously change, for example, from 0 to 1. Physically, this means that we are assuming that the Y-forces can instantaneously shift fires as desired.

When one considers command and control problems in combat, the above implicit assumption on ϕ (instantaneous jumps permitted) does not seem to be a realistic one. A better assumption appears to be that there is a limit to how fast ϕ can be changed. For example, consider a

homogeneous Y-force in combat against heterogeneous enemy forces. A command and control system directs the fire of the homogeneous Y-force against particular enemy targets. The effectiveness of the command and control system might be measured in terms of the speed and accuracy with which units of the Y-force react to orders as to which type of enemy unit at which to fire (see [13] for some similar ideas). This fire distribution process may be described in terms of a distribution of fire variable ϕ. We have thus been led to consider target selection problems in which the rate of change of the allocation variable is bounded (i.e. instantaneous shifts in fire are not allowed). For reasons discussed below, we have chosen to call such a situation "inertial combat."

Although problems in which curves are restricted to lie in a given domain were considered in the classical calculus of variations as long ago as 1831 [5] (see also [1]) and discussed by Weierstrass in his lectures of 1879 (see p. 395 of [1]), development of optimality conditions for optimal control problems with state variable inequality constraints has been accomplished only comparatively recently. As the author pointed out in [19], state variable inequality constraints (SVIC's) are present in all Lanchester-type optimal control/differential game problems. Recent activity in developing necessary conditions of optimality for problems with SVIC's apparently owes its origin to the work of Gamkrelidze (for an English translation of his original work see Chapter VI of [11]). Gamkrelidze points out that in many physical problems there are restrictions not only on the control parameters but also on the state (phase) space. He (see p. 263 of [11]) refers to piecewise continuous controls as "inertialess controls," since such controls can, if need by, instantaneously jump from

one value to another. Following Gamkrelidze then, we use the term inertial combat to refer to a target selection problem in which the rate of change of the distribution of fire is bounded.

In [17] the author first applied the theory of SVIC's (the approach of Gamkrelidze [11]) to an allocation problem in the Lanchester theory of combat. In this first application little more than developing optimality conditions on constrained subarcs was done. In [19] we introduced a more convenient approach to first order SVIC's (the approach of Speyer (see [6])) and used theory (including corner conditions and boundary conditions for the adjoint variables (see also [21])) to completely solve a problem similar to (1). The paper at hand further extends such results: we consider a problem with a second order* SVIC (as well as first order SVIC's). This application is possible because of theoretical results recently obtained by the author who extended Gamkrelidze's multiplier condition [11] (see also [20]) to a $p^{\underline{th}}$ order SVIC [22] (see also [21]). The reader can find a further discussion of the theory of SVIC's in [17] and [19] (see also [6] and [10]).

This paper is organized in the following fashion. First, we discuss the optimal control problem (optimal fire distribution in the presence of command and control limitations). Then, the basic necessary conditions of optimality are developed for the problem. Next, the synthesis of the extremal fire distribution policy is outlined in several cases. The determination of the optimal fire distribution policy is discussed, and

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*This terminology was apparently first coined by Bryson, Denham, and Dreyfus [3]. They say that a problem has a $p^{\underline{th}}$ order SVIC when the $p^{\underline{th}}$ time (total) derivative of the state-variable constraint is the first derivative to explicitly contain the control variables.

a more general model considered. Finally, we discuss the insights gained into the optimization of combat dynamics from our study of the problem at hand.

2. The Fire Distribution Problem in the Presence of Command and Control Limitations.

Accordingly, we consider the following problem:

$$\underset{u(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\} \text{ with } T_1 \text{ specified,}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2,$$

$$\frac{d\phi}{dt} = u, \tag{2}$$

$$x_1, x_2, y \geq 0, \quad T \leq T_1, \quad 0 \leq \phi \leq 1, \quad \text{and} \quad -R_L \leq u \leq R_U,$$

and with initial conditions

$$x_1(t=0) = x_1^o, \quad x_2(t=0) = x_2^o, \quad y(t=0) = y_0, \quad \phi(t=0) = \phi_0,$$

where all symbols are defined in the next section and $R_L, R_U > 0$. It should be noted in the above model it is no longer possible for ϕ to instantaneously change from, for example, 0 to 1 as it had been for (1). As we discussed in the introduction, this is how we incorporate command and control limitations into our model.

We will focus primarily upon the development of the basic necessary conditions of optimality for (2) and the synthesis of extremal control from

these. As discussed more thoroughly in Section 8, it is not practical for computational reasons to completely carry out the determination of the optimal control. Thus, for this more limited goal of characterizing the optimal fire distribution policy the nature of the planning horizon (terminal target set or prescribed duration) doesn't make any difference, and we will be purposely vague on this point.*

The reader should note that the control variable in problem (2) is u, while ϕ (the control variable in problem (1)) is now a state variable. Hence, the restriction $0 \leq \phi \leq 1$ is now equivalent to two (first order) SVIC's. When we use the approach of Gamkrelidze (see Chapter VI of [11]) (as modified by Bryson et al. [3] (see also [6])), a SVIC such as $C(t,x_i(t)) \leq 0$ is replaced by the point constraint

$$C(t_{entry},x_i) = 0,$$

and the control inequality constraint on the state boundary ($C=0$)

$$\frac{dC}{dt}(t,x_i,u) \leq 0 \quad \text{for} \quad t \in [t_{entry},t_{exit}].$$

Thus, for $\phi - 1 \leq 0$, we treat boundary arcs when $\phi = 1$ by considering $\phi(t_{entry}) = 1$ and then requiring

$$\frac{d\phi}{dt} = u \leq 0 \quad \text{for} \quad t \in [t_{entry},t_{exit}] \quad \text{when} \quad \phi - 1 = 0, \tag{3}$$

and for $-\phi \leq 0$, we treat boundary arcs when $\phi = 0$ by considering $\phi(t_{entry}) = 0$ and then requiring

$$-\frac{d\phi}{dt} = -u \leq 0 \quad \text{for} \quad t \in [t_{entry},t_{exit}] \quad \text{when} \quad -\phi = 0. \tag{4}$$

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*However, see [14]-[18] for the type of considerations (i.e. enumeration of all possible terminal states) required for developing a complete solution to such a problem.

To avoid being encumbered by too many symbols, we will consider only one of the two SVIC's $x_1, x_2 \geq 0$. Clearly, we lose no generality in considering $x_1 \geq 0$. In this case, we have

$$C(t,x_i) = - x_1 \leq 0, \tag{5}$$

$$\frac{dC}{dt}(t,x_i) = - \frac{dx_1}{dt} = \phi a_1 y, \tag{6}$$

$$\frac{d^2C}{dt^2}(t,x_i,u) = ua_1 y - \phi a_1(b_1x_1+b_2x_2). \tag{7}$$

On a constrainted subarc on which $x_1(t) = 0$ for $t_{entry} \leq t \leq t_{exit}$, the SVIC is replaced by the point constraints

$$C(t_{entry},x_i) = - x_1(t_{entry}) = 0, \tag{8}$$

and

$$\frac{dC}{dt}(t_{entry},x_i) = - \frac{dx_1}{dt}(t_{entry}) = \phi(t_{entry})a_1 y = 0, \tag{9}$$

and the control inequality constraint

$$\frac{d^2C}{dt^2}(t,x_i,u) = ua_1 y - \phi a_1(b_1x_1+b_2x_2) \leq 0 \quad \text{for} \quad t \in [t_{entry},t_{exit}]. \tag{10}$$

Thus, $x_1 \geq 0$ is a second order SVIC. Clearly, when $x_1(t) = 0$ for a finite interval of time, by (6) we must have $\phi^*(t) = 0$ (since $y > 0$) and then (7) yields $u^*(t) = 0$, where we have considered the state equations (2).

3. Notation.

The symbols which are used in this paper are defined as follows:

a_1,a_2,b_1,b_2 = constant attrition-rate coefficients,

$(\dot{C})(t_e) = \left.\frac{dC}{dt}\right|_{t=t_e}$,

$F(\phi,u)$ = rate of change of fraction of Y-fire directed at X_1 in more general redistribution of fire model,

H = Hamiltonian function,

J = criterion (return) functional = $ry(T) - px_1(T) - qx_2(T)$,

p,q,r = utilities assigned to surviving X_1, X_2 and Y forces respectively,

$p_i(t)$ for i = 1,2,3,4 = dual variable corresponding to $x_i(t)$ ($x_3(t) = y(t)$, $x_4(t) = \phi(t)$),

R_L, R_U = lower and upper bounds on magnitude of rate of change of (i.e. $-R_L \le u \le R_U$),

t = time after beginning of battle,

t_e = time of entry to constrained subarc,

$t_e^- = \lim_{\substack{t \to t_e \\ t \le t_e}} t$,

t_e^o = time of entry to constrained subarc with $\phi(t) = 0$ for $t_e^o \le t \le t_\ell^o$ (similarly for t_e^1),

t_ℓ = time of leaving constrained subarc,

t_s = time at which $u^*(t)$ switches from R_U to $-R_L$ with $0 < \phi < 1$; defined by $p_4(t=t_s) = 0$,

t_1 = time at which X_1 is annihilated, i.e. $x_1(t_1) = 0$,

T = time at which battle ends,

T_1 = maximum possible duration for battle, i.e. $T \le T_1$,

u = control variable for redistribution of fire,

$v = v(t) = a_1(-p_1(t)) - a_2(-p_2(t))$,

W = Bellman's optimal value function,

x_1, x_2, y = combatant force levels; with initial values x_1^o, x_2^o, y_0,

$\delta, \delta_1, \delta_2$ = positive constants,

$\eta_1(t)(\eta_2(t))$ = multiplier corresponding to state variable inequality constraint $\phi \le 1$ $(\phi \ge 0)$,

$\mu(t)$ = multiplier corresponding to state variable inequality constraint $x_1 \ge 0$,

ν_0 = multiplier corresponding to intermediate equality constraint $x_1(t_e) = 0$ used to (formally) handle entry to a constrained subarc with $x_1(t) = 0$.

ν_i for $i = 1,2,3$ = multiplier corresponding to state variable terminal inequality constraint $x_i(T) \ge 0$ $(x_3(T) = y(T))$,

$\nu_4(\nu_5)$ = multiplier corresponding to state variable terminal inequality constraint $\phi(T) \le 1$ $(\phi(T) \ge 0)$,

ϕ = fraction of Y-fire directed at X_1,

τ = "backwards time" from the end of the battle; defined by $\tau = T - t$, i.e. the time remaining before the end of battle,

τ_1 = "backwards time" of the first change in the sign of the switching function v, i.e. $v(t=T-\tau_1) = 0$.

4. Characterization of an Optimal Fire Distribution Policy.

Using Gamkrelidze's approach and considering (3), (4), (8), (9), and (10), we have that the Hamiltonian is given by [2], [6], [11]

$$H(t,x_i,p_i,u) = -p_1\phi a_1 y - p_2(1-\phi)a_2 y - p_3(b_1x_1+b_2x_2) + p_4 u$$
$$- \eta_1(t)u + \eta_2(t)u - \mu(t)\{ua_1y - \phi a_1(b_1x_1+b_2x_2)\}, \qquad (11)$$

where

$$\eta_1(t)\begin{cases} = 0 & \text{for } \phi < 1, \\ \geq 0 & \text{for } \phi = 1, \end{cases} \qquad \eta_2(t)\begin{cases} = 0 & \text{for } \phi > 0, \\ \geq 0 & \text{for } \phi = 0, \end{cases}$$

and

$$\mu(t)\begin{cases} = 0 & \text{for } x_1 > 0, \\ \geq 0 & \text{for } x_1 = 0. \end{cases}$$

We have adopted above the following correspondence between state and dual variables:

state variable	dual variable
x_1	p_1,
x_2	p_2,
y	p_3,
ϕ	p_4.

Again, to avoid being encumbered with too many symbols, we have only considered one (i.e. $x_1 \geq 0$) of the two SVIC's $x_1, x_2 \geq 0$. The other SVIC (i.e. $x_2 \geq 0$) is handled in a similar way. The adjoint system of differential equations for the dual variables is

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x_1} = b_1 p_3 - \mu(t)\phi a_1 b_1, \tag{12}$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial x_2} = b_2 p_3 - \mu(t)\phi a_1 b_2, \tag{13}$$

$$\frac{dp_3}{dt} = -\frac{\partial H}{\partial y} = \phi a_1 p_1 + (1-\phi)a_2 p_2 + \mu(t)a_1 u^*, \tag{14}$$

$$\frac{dp_4}{dt} = -\frac{\partial H}{\partial \phi} = (a_1 p_1 - a_2 p_2)y - \mu(t)a_1(b_1 x_1 + b_2 x_2). \tag{15}$$

The boundary conditions at $t = T$ for the adjoint (or dual) variables may be written

$$p_1(t=T) = -p + \nu_1, \quad p_2(t=T) = -q + \nu_2, \quad p_3(t=T) = r + \nu_3, \quad p_4(t=T) = \nu_4 - \nu_5,$$

where ν_i for $i = 1,\ldots,5$ are undetermined multipliers which require the specification of additional information to be further delineated. To this end,* let us consider the case in which Y loses with $T < T_1$ (i.e. T defined by $y(T) = 0$). From the transversality condition $H(T,x_i(T), p_i(T),u^*(T)) = 0$, we obtain $p_3(T) = 0$. Then, we have [2], [21]

$$p_1(T) = -p + \nu_1, \quad p_2(T) = -q + \nu_2, \quad p_3(T) = 0, \quad p_4(T) = \nu_4 - \nu_5, \tag{16}$$

where

$$\text{for } i = 1,2 \quad \nu_i \begin{cases} = 0 & \text{for } x_i(T) > 0, \\ \geq 0 & \text{for } x_i(T) = 0 \text{ but } x_i(t) > 0 \text{ for } t < T, \\ \text{unrestricted} & \text{when } x_i(t) = 0 \text{ for } t_i \leq t \leq T \\ & \quad \text{with } t_i < T, \end{cases}$$

and

$$\nu_4 \begin{cases} = 0 & \text{for } \phi(T) > 0, \\ \geq 0 & \text{for } \phi(T) = 0, \end{cases} \qquad \nu_5 \begin{cases} = 0 & \text{for } \phi(T) < 1, \\ \geq 0 & \text{for } \phi(T) = 1. \end{cases}$$

When $x_1,x_2,y > 0$ and $0 < \phi < 1$, the (extremal) control law is determined by the maximum principle. Hence, we consider

$$\underset{-R_L \leq u \leq R_U}{\text{maximize}} \; H(t,x_i,p_i,u),$$

and this yields

$$u^*(t) = \begin{cases} R_U & \text{for } p_4(t) > 0, \\ -R_L & \text{for } p_4(t) < 0. \end{cases} \tag{17}$$

---

*Other cases are handled in a similar manner. See [19] for a problem in which the boundary conditions for the adjoint variables are worked out for all the battle's end states.

We must further investigate the possibility of singular subarcs [7] (also see Chapter 8 in [2]) on which $\frac{\partial H}{\partial u} = 0$ for a finite interval of time (so that all its time derivatives vanish). The condition that $\frac{\partial H}{\partial u} = 0$ yields that on a singular subarc we must have

$$p_4(t) = 0. \tag{18}$$

The condition $\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0$ then yields

$$a_1 p_1(t) = a_2 p_2(t). \tag{19}$$

Proceeding to the next time derivative, we would have on a singular subarc on which (18) and (19) hold that

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = \gamma p_3(a_1 b_1 - a_2 b_2). \tag{20}$$

From (20), we see that a singular solution is impossible, since it is impossible (in general) to have $\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = 0$ for a finite interval of time.

4.1. Necessary Conditions of Optimality on Constrained Subarc for ϕ.

On a constrained subarc on which $\phi(t) = 1$ for $t_e \le t \le t_\ell$ the control is determined by $\frac{d\phi}{dt} = 0$ and hence

$$u^*(t) = 0 \quad \text{for} \quad t_1 < t < t_2. \tag{21}$$

The multiplier $\eta_1(t)$ is determined by the condition $\frac{\partial H}{\partial u} = 0$ and hence

$$\eta_1(t) = p_4(t). \tag{22}$$

The requirement that $\eta_1(t) \ge 0$ yields that on the constrained subarc we must have

$$p_4(t) \geq 0. \tag{23}$$

Differentiating (23) and combining with (15), we obtain

$$\dot{\eta}_1(t) = y(a_1p_1 - a_2p_2), \tag{24}$$

so that Gamkrelidze's condition $\dot{\eta}_1(t) \leq 0$ [11] (see also [20]) is only satisfied on a constrained subarc with $\phi = 1$ when

$$a_1(-p_1(t)) \geq a_2(-p(t)), \tag{25}$$

which the reader will, of course, recognize as a result for the corresponding "inertialess" combat problem (see [14], [19]). Denoting the time of an entrance corner by t_e and that of an exit corner by t_ℓ, the corner conditions (see [17] and [19] for further discussion, especially for corners interior to the state space (which are not explicitly discussed here)) yield that at an entrance corner we have [10]

$$p_i(t_e^-) = p_i(t_e^+) \quad \text{for} \quad i = 1,2,3, \tag{26}$$

and

$$p_4(t_e^-) = 0 = p_4(t_e^+) - \eta_1(t_e^+), \tag{27}$$

or

$$p_4(t_e^+) = \eta_1(t_e^+), \tag{28}$$

where t_e^- denotes a left-hand limit. The reader should note that (28) is in consonance with (22). Furthermore, at an exit corner we have

$$p_i(t_\ell^-) = p_i(t_\ell^+) \quad \text{for} \quad i = 1,2,3,4. \tag{29}$$

Considering either $\eta_1(t_\ell^-) = 0$ or $H(t_\ell^-) = H(t_\ell^+)$, we find that

$$p_4(t_\ell^-) = 0 = p_4(t_\ell^+). \tag{30}$$

Considering (24), (28), and again $\eta_1(t_\ell^-) = 0$, we see that when there is an exit from the constrained subarc at $t = t_\ell$, then (28) yields

$$p_4(t_e^+) = \eta_1(t_e^+) = \int_{t_e}^{t_\ell} y(t)\{a_2p_2(t) - a_1p_1(t)\}dt \geq 0. \tag{31}$$

On a constrained subarc on which $\phi(t) = 0$ for $t_1 \leq t \leq t_2$ the details are similar to the above with the control again given by (21), since again $\frac{d\phi}{dt} = 0$. Determinations similar to the above yield

$$\eta_2(t) = -p_4(t), \tag{32}$$

and that it is necessary on the constrained subarc that

$$p_4(t) \leq 0, \tag{33}$$

and

$$a_1(-p_1(t)) \leq a_2(-p_2(t)), \tag{34}$$

the latter condition (34) being a consequence of Gamkrelidze's multiplier condition $\dot{\eta}_2(t) \leq 0$. Corner conditions similar to (26) through (30) also must hold. When there is an exit from the constrained subarc at $t = t_\ell$, we find that

$$p_4(t_e^+) = -\eta_2(t_e^+) = \int_{t_e}^{t_\ell} y(t)\{a_2p_2(t) - a_1p_1(t)\}dt \leq 0. \tag{35}$$

4.2. Necessary Conditions of Optimality on Constrained Subarc for x_1.

On a constrained subarc on which $x_1(t) = 0$ for $t_e \leq t \leq T$ the control is determined by $\frac{d^2x_1}{dt^2} = 0$ which (along with the requirement that $\frac{dx_1}{dt} = 0$) yields

$$u^*(t) = 0 \quad \text{for} \quad t_1 < t \leq T, \tag{36}$$

and

$$\phi(t) = 0 \quad \text{for} \quad t_1 \leq t \leq T. \tag{37}$$

The multiplier $\mu(t)$ is determined by the condition $\frac{\partial H}{\partial u} = 0$ and hence

$$p_4(t) = \mu(t)a_1y = 0, \tag{38}$$

or

$$\mu(t) = \frac{p_4(t)}{a_1y}. \tag{39}$$

Differentiating (38) with respect to time and combining with (15), we obtain*

$$\dot{\mu}(t) = \frac{1}{a_1}(a_1p_1 - a_2p_2). \tag{40}$$

Further differentiation and combination with (12), (13), and the condition $\phi(t) = 0$ yields that

$$\ddot{\mu}(t) = \frac{p_3(t)}{a_1}(a_1b_1 - a_2b_2). \tag{41}$$

The necessary condition of optimality on a constrained subarc of a second order SVIC [22] is that $(-1)^k \frac{d^k\mu}{dt^k} \geq 0$ for $k = 0,1,2,$ and hence considering (39), (40), and (41) we must have

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*This result was obtained after cancellation of a term $y(t)$. Hence, a different argument is required when $y(t) = 0$. This latter condition, however, only occurs at most at a single isolated point $t = T$. See [19] for a similar occurrence and further discussion.

$$p_4(t) \geq 0, \tag{42}$$

$$a_1(-p_1(t)) \geq a_2(-p_2(t)), \tag{43}$$

and

$$a_1 b_1 \geq a_2 b_2, \tag{44}$$

since it is readily shown that $p_3(t) > 0$ for $t < T$. It is of interest to note that (43) and (44) have previously been shown [17] to be necessary conditions of optimality for having $x_1 = 0$ for a finite interval of time in the inertialless combat problem (1).

As in [14], [19] let us make the nonrestrictive assumption that $\underline{a_1 b_1 > a_2 b_2}$. This then implies that it is non-optimal to have $x_2 = 0$ for a finite interval of time, since (44) must hold with the sense of the inequality reversed on such a constrained subarc.

For a second order SVIC $C(t,x_1) \leq 0$, we must have at an entrance corner to a constrained subarc [10]

$$p_i(t_e^-) = p_i(t_e^+) - \nu_0 \frac{\partial C}{\partial x_i}(t_e) - \mu(t_e^+) \frac{\partial}{\partial x_i}\{(\dot{C})(t_e)\} \quad \text{for } i = 1,\ldots,n, \tag{45}$$

and

$$H(t_e^-) = H(t_e^+) + \nu_0 \frac{\partial C}{\partial t}(t_e) + \mu(t_e^+) \frac{\partial}{\partial t}\{(\dot{C})(t_e)\}, \tag{46}$$

where [6] $\nu_0 \geq 0$. Recalling (5) and (6), we find for the problem at hand at an entrance at $t = t_e$ to a constrained subarc on which $x_1 = 0$ that the following hold

$$p_1(t_e^-) = p_1(t_e^+) + \nu_0, \tag{47}$$

$$p_i(t_e^-) = p_i(t_e^+) \quad \text{for } i = 2,3, \tag{48}$$

$$p_4(t_e^-) = p_4(t_e^+) - \mu(t_e^+) a_1 y. \tag{49}$$

From (38) we have

$$p_4(t_e^+) = \mu(t_e^+)a_1 y, \tag{50}$$

so that (49) yields

$$p_4(t_e^-) = 0. \tag{51}$$

Furthermore, the corner condition (46) (which reads $H(t_e^-) = H(t_e^+)$) is satisfied when (48) and (49) hold. It should be clear that no trajectory can leave a constrained subarc on which $x_1 = 0$ after entry. Hence, we omit discussion of the corner conditions at exit corners.

It will now be shown that we must have $\nu_0 > 0$. Considering the nonrestrictive assumption $a_1 b_1 > a_2 b_2$, it should be clear that we must have $x_1, x_2, y, \phi > 0$ for $0 < t < t_e$ (we might have $\phi(t=0) = 0$). Then $\phi(t) > 0$ for $0 < t < t_e$ and $\phi(t) = 0$ for $t_e \leq t$ with $\phi(t)$ being continuous imply that $u^*(t_e^-) = -R_L$, and thus by (17) and (51)

$$p_4(t) < 0 \quad \text{for} \quad t \in (t_e - \delta, t_e) \quad \text{with} \quad p_4(t_e^-) = 0, \tag{52}$$

where $\delta > 0$ is a suitably chosen constant, and we have used the fact that it is impossible for $p_4(t) = 0$ for a finite interval of time (this was established when we showed the impossibility of a singular solution). Expanding $p_4(t)$ for $t < t_e$ in a Taylor series about $t = t_e^-$, we find that

$$(t_e - t)\frac{dp_4}{dt}(\tilde{t}) = (-p_4(t)) > 0, \tag{53}$$

where $\tilde{t} \in (t, t_e)$. Hence, $\frac{dp_4}{dt}(t) > 0$ for all $t \in (t_e - \delta_1, t_e)$ where $0 < \delta_1 \leq \delta$. Recalling (15) and that $y > 0$, we have

$$a_1 p_1(t) > a_2 p_2(t) \quad \text{for} \quad t \in (t_e - \delta_1, t_e), \tag{54}$$

so that in the limit we have

$$a_1p_1(t_e^-) \geq a_2p_2(t_e^-). \tag{55}$$

Next, we show that we must have

$$a_1p_1(t_e^-) > a_2p_2(t_e^-). \tag{56}$$

The proof is by contradiction. Considering (55), we assume that $a_1p_1(t_e^-) = a_2p_2(t_e^-)$. Again expand $p_4(t)$ for $t < t_e$ in a Taylor series about $t = t_e^-$. Recalling (15) and using the above assumption and (51), we find that for $t < t_e$

$$p_4(t) = \frac{(t_e-t)^2}{2}\frac{d^2p_4}{dt^2}(\bar{t}), \tag{57}$$

where $\bar{t} \in (t,t_e)$. Using the state and adjoint equations, we readily compute that

$$\frac{d^2p_4}{dt^2}(t) = yp_3(t)(a_1b_1-a_2b_2) - \{a_1p_1(t)-a_2p_2(t)\}(b_1x_1+b_2x_2). \tag{58}$$

By the continuity of the dual variables between corners, t can be chosen $\in(t_e-\delta,t_e)$ such that for all $\tilde{t} \in (t,t_e)$ we have

$$\frac{d^2p_4}{dt^2}(\tilde{t}) > 0,$$

and hence by (57) we have a contradiction to (52). Thus, (56) must hold.

Next, we show that we must have

$$a_1p_1(t_e^+) < a_2p_2(t_e^+). \tag{59}$$

This follows immediately from $p_3(t) > 0$ for $t < T \Rightarrow \ddot{\mu}(t) > 0$ for $t_e \leq t < T$. Then $\dot{\mu}(t=T) \leq 0 \Rightarrow \mu(t) < 0$ for $t_e < t < T$, and reference to (40) yields the desired result (59).

The proof that $\nu_0 > 0$ now readily follows. First, we observe that (48), (56), and (59) yield

$$a_1p_1(t_e^-) > a_2p_2(t_e^-) = a_2p_2(t_e^+) > a_1p_1(t_e^+), \tag{60}$$

or

$$p_1(t_e^-) > p_1(t_e^+), \tag{61}$$

whence follows $\nu_0 > 0$ by (47). Moreover, ν_0 is chosen so that $\phi = 0$ precisely when $x_1(t_e) = 0.$*

5. <u>Synthesis of Extremal Policy when $x_1(T), x_2(T) > 0$.</u>

In this and the next two sections we synthesize the extremal fire distribution policy for all cases in which Y loses with $T < T_1$** (the same case for which the boundary conditions for the adjoint variables were given in Section 4). By the synthesis of the extremal control we mean the explicit determination (using the necessary conditions of optimality) of the time history of the extremal control from initial to terminal time (see [17]-[19] for further, more extensive discussion).

The basic idea is to trace extremals*** backwards from a given terminal state in such a way as to guarantee the satisfaction of the initial

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*This multiplier ν_0 arises because the system loses two degrees of freedom when it enters the constrained subarc (see pp. 411-412 of [10]). For a first order SVIC, the value of the multiplier $\mu(t_e^+)$ at the entrance corner accounts for the loss of one degree of freedom by the system upon entering the constrained subarc. For a second order SVIC, there are two degrees of freedom lost this way (for the problem at hand, $x_1 = 0$ and $\phi = 0$ on the constrained subarc), one of which is accounted for by $\mu(t_e^+)$.

**Other cases are handled in a similar manner. See [18] and [19] for problems in which this is done for every end state of battle.

***By an extremal we mean a path (or trajectory) on which the necessary conditions of optimality are satisfied at every point in time.

conditions. Thus, it is convenient to introduce the "backwards time" variable τ defined by $\tau = T - t$. We observe that $\frac{d}{dt} = -\frac{d}{d\tau}$ but $\frac{d^2}{dt^2} = \frac{d^2}{d\tau^2}$. It is also convenient to define

$$v(t) = a_1(-p_1(t)) - a_2(-p_2(t)), \tag{62}$$

so that differentiation and combination with (12) and (13) yield

$$\frac{dv}{dt} = -p_3(t)(a_1b_1 - a_2b_2). \tag{63}$$

Then our nonrestrictive assumption that $a_1b_1 > a_2b_2$ yields that

$$\frac{dv}{dt}(t) < 0 \quad \text{for all} \quad t < T, \tag{64}$$

since it is easily shown that $p_3(t) > 0$ for $t < T$. Using (62), it is convenient to write (15) (for $x_1 > 0$) as

$$\frac{dp_4}{dt} = -yv, \tag{65}$$

and hence

$$\frac{d^2p_4}{dt^2} = (b_1x_1 + b_2x_2)v - y\frac{dv}{dt}. \tag{66}$$

In synthesizing an extremal there are two cases to consider:

Case (a) $a_1p \geq a_2q$,

Case (b) $a_1p < a_2q$.

For Case (a): $a_1p \geq a_2q$, it is convenient to first observe that a Taylor series expansion of $p_4(\tau)$ about $\tau = 0$ yields for $\tau \geq 0$

$$p_4(\tau) = p_4(\tau{=}0) + \tau\frac{dp_4}{d\tau}(\tau{=}0) + \frac{\tau^2}{2}\frac{d^2p_4}{d\tau^2}(\tau{=}\tilde{\tau}), \tag{67}$$

where $\tilde{\tau} \in (0,\tau)$. In this case we have

$$v(\tau=0) = a_1 p - a_2 q \geq 0, \tag{68}$$

so that considering (64) it is readily seen that

$$v(\tau) > 0 \text{ for } \tau > 0, \tag{69}$$

and hence (65) and (66) yield

$$\frac{dp_4}{d\tau}(\tau=0) = y(T)(a_1 p - a_2 q) = 0, \tag{70}$$

since $y(T) = 0$ and

$$\frac{d^2 p_4}{d\tau^2}(\tau) > 0 \text{ for } \tau > 0. \tag{71}$$

Furthermore, there are three subcases to be considered when $a_1 p \geq a_2 q$:

Subcase (a1) $\phi(t=T) = 0$,

Subcase (a2) $0 < \phi(t=T) < 1$,

Subcase (a3) $\phi(t=T) = 1$.

We will now show that Subcase (a1) is not consistent with an optimal policy and work out details for the other two cases.

Subcase (a1): $\phi(t=T) = 0$ when $a_1 p \geq a_2 q$.

Since $\phi(t=T) = 0$, (16) yields that $p_4(\tau=0) = \nu_4 \geq 0$. Then (67), (70), and (71) yield that

$$p_4(\tau) > 0 \text{ for } \tau > 0. \tag{72}$$

If the system would be on a constrained subarc for a finite interval of time, i.e. $\phi(t) = 0$ for $t_e \leq t \leq T$, then (72) is a violation of the necessary condition of optimality (33). (We also note that Gamkrelidze's

necessary condition $\dot{\eta}_2(t) \le 0$ is violated, since by (32), (65) and (69) we have $\dot{\eta}_2(t) = \gamma v(t) > 0$ for $t_e \le t < T$.) If we were not on a constrained subarc for a finite interval of time, then we must have $\phi(\tau) > 0$ for $0 < \tau < \delta$ with $\phi(\tau=0) = 0$. This implies that $u^*(\tau) = -R_L$ for $0 \le \tau < \delta_1 \le \delta$, and this is impossible by (17) and (72). Hence this case is inconsistent with an optimal policy.

<u>Subcase (a2): $0 < \phi(t=T) < 1$ when $a_1 p \ge a_2 q$.</u>

Since $0 < \phi(t=T) < 1$, (16) yields $p_4(\tau=0) = 0$, and (72) again follows. Then by (17) and (72) we have

$$u^*(\tau) = R_U \quad \text{for} \quad 0 \le \tau \le T. \tag{73}$$

Denoting $\phi(t=0)$ by ϕ_0, we then have $\phi_0 + R_U T = \phi(t=T) < 1$, so that this case happens when $T < (1-\phi_0)/R_U$. The extremal policy is then given by

$$u^*(t) = R_U \quad \text{for} \quad 0 \le t \le T < (1-\phi_0)/R_U. \tag{74}$$

For longer times we must go to the next subcase.

<u>Subcase (a3): $\phi(t=T) = 1$ when $a_1 p \ge a_2 q$.</u>

Since $\phi(t=T) = 1$, (16) now yields $p_4(\tau=0) = -\nu_5 \le 0$. It may be shown that a contradiction arises unless $\nu_5 = 0$. Hence, we must have $p_4(\tau=0) = 0$. If $\phi(t) < 1$ for $T - \delta \le t < T$ where $\delta > 0$, then the development of the previous subcase holds. If we are on a constrained subarc with $\phi(t) = 1$ for $t_e \le t \le T$, then by (24) and (62) we have $\dot{\eta}_1(t) = -\gamma v(t)$. Hence Gamkrelidze's necessary condition $\dot{\eta}_1(t) \le 0$ is satisfied by (69). Thus, we can remain (in backwards progression at the end $t = T$) on the constrained subarc until we have to get off to meet the initial condition $\phi(t=0) = \phi_0$. As we work backwards and leave the

constrained subarc (but in forwards time enter at t_e), the corner condition (27) and a Taylor series expansion of $p_4(t)$ about $t = t_e$ yield for $0 \le t \le t_e$

$$p_4(t) = -(t_e - t)\,\frac{dp_4}{dt}\,(t=t_e) + \frac{(t_e - t)^2}{2}\,\frac{d^2p_4}{d\tau^2}\,(t=\tilde{t}), \tag{75}$$

where $\tilde{t} \in (t, t_e)$. Recalling (65), (69), and (71), we see that $p_4(t) \ge 0$ for $0 \le t \le t_e$ so that the rest of the analysis is similar to that of the preceding subcase. Hence

when $T \ge (1-\phi_0)/R_U$, we have

$$\begin{aligned} &\text{for } 0 \le t \le (1-\phi_0)/R_U, \quad u^*(t) = R_U \text{ and } \phi^*(t) = \phi_0 + R_U t, \\ &\text{for } (1-\phi_0)/R_U < t \le T, \quad u^*(t) = 0 \text{ and } \phi^*(t) = 1. \end{aligned} \tag{76}$$

<u>For Case (b)</u>: <u>$a_1 p < a_2 q$</u>, we have that (recalling (62))

$$v(\tau=0) = a_1 p - a_2 q < 0. \tag{77}$$

From (64) we have $\frac{dv}{d\tau}(\tau) > 0$ for $\tau > 0$ with $v(\tau)$ a continuous function (see [2] and above corner conditions at boundary of state space (26)), and thus at some (backwards) time $v(\tau)$ must become zero. Denote this "backwards time" as τ_1. Thus $v(\tau=\tau_1) = 0$. There are, again, three subcases to be considered when $a_1 p < a_2 q$:

Subcase (b1) $\phi(t=T) = 0$,

Subcase (b2) $0 < \phi(t=T) < 1$,

Subcase (b3) $\phi(t=T) = 1$.

Analysis of these subcases is similar to that given for Case (a) with Subcase (b3) being impossible.

Let us now observe (recalling (25) and (34)) that in order to satisfy Gamkrelidze's condition on constrained subarcs with $\phi = 0$ or $\phi = 1$, we must have

$$v(t) \geq 0 \text{ when } \phi(t) = 1 \text{ for a finite interval of time,} \tag{78}$$

and

$$v(t) \leq 0 \text{ when } \phi(t) = 0 \text{ for a finite interval of time.} \tag{79}$$

We have previously noted in Section 4.1 the correspondence of these results to those for "inertialless" combat. We now consider the case when $\phi(t=T) = 0$. Let t_e^o denote the (forward) time when the system enters a constrained subarc with $\phi = 0$; similarly t_e^1 denotes the time of leaving one with $\phi = 1$. We further assume that

$$\phi(t) = 0 \quad \text{for} \quad t_e^o \leq t \leq T. \tag{80}$$

We now prove that it is impossible to have $v(t=t_e^o) = 0$; in fact, $v(t)$ must be < 0 <u>before</u> $\phi = 0$. We begin by observing that $\phi(t=t_e^o) = 0$ with $0 < \phi(t) < 1$ for $t_e^o - \delta < t < t_e^o$ where $\delta > 0$. Hence, $u^*(t) = -R_L$ and $p_4(t) \leq 0$ by (17) for $t_e^o - \delta_1 < t < t_e^o$ where $0 < \delta_1 \leq \delta$. Considering a Taylor series expansion about $t = t_e^{o-}$ and (27), (65), and (66), we have for $t \leq t_e^o$

$$p_4(t) = (t_e^o - t)y(t_e^o)v(t_e^o) + \frac{(t_e^o - t)^2}{2}\{(b_1x_1 + b_2x_2)v(\tilde{t}) - y(\tilde{t})\frac{dv}{dt}(\tilde{t})\}, \tag{81}$$

where $\tilde{t} \in (t, t_e^o)$. Considering that $\tilde{t} < t_e^o$, (64), and (81), it is easily seen that $v(t_e^o) \geq 0 \Rightarrow p_4(t) > 0$ for $t < t_e^o$, which is impossible by the above. Hence,

$$v(t=t_e^o) < 0. \tag{82}$$

Now, it is readily shown that for $t_\ell^1 \le t \le t_e^o$, $p_4(t) = \int_t^{t_e^o} y(s)v(s)ds$ so that recalling (30)

$$p_4(t_\ell^{1+}) = \int_{t_\ell^1}^{t_e^o} y(s)v(s)ds. \tag{83}$$

Then, the continuity of $v(t)$, (64) and (82) yield that $v(t_\ell^1) > 0$. Denote the time at which $v = 0$ as $T - \tau_1$ so that $v(t{=}T-\tau_1) = 0$. Then since we also have $p_4(t) = \int_{t_\ell^1}^t y(s)v(s)ds$, it follows that $p_4(t) < 0$ for $t_\ell^1 < t < t_e^o$ and hence by (17)

$$\dot{u}^*(t) = -R_L \quad \text{for} \quad t_\ell^1 < t < t_e^o.$$

It follows that $t_e^o - t_\ell^1 = 1/R_L$. The times t_ℓ^1 and t_e^o are determined by the conditions

$$T - \tau_1 \in (t_\ell^1, t_e^o) \quad \text{where} \quad v(t{=}T-\tau_1) = 0, \tag{84}$$

and

$$\int_{t_\ell^1}^{t_e^o} y(s)v(s)ds = 0, \tag{85}$$

which may be written as

$$\int_{t_\ell^1}^{T-\tau_1} y(t)v(t)dt = -\int_{T-\tau_1}^{t_e^o} y(t)v(t)dt. \tag{86}$$

The relationship of the times t_ℓ^1, $T-\tau_1$, and t_e^o to the time histories of $\phi(t)$ and $v(t)$ are shown in Figure 1.

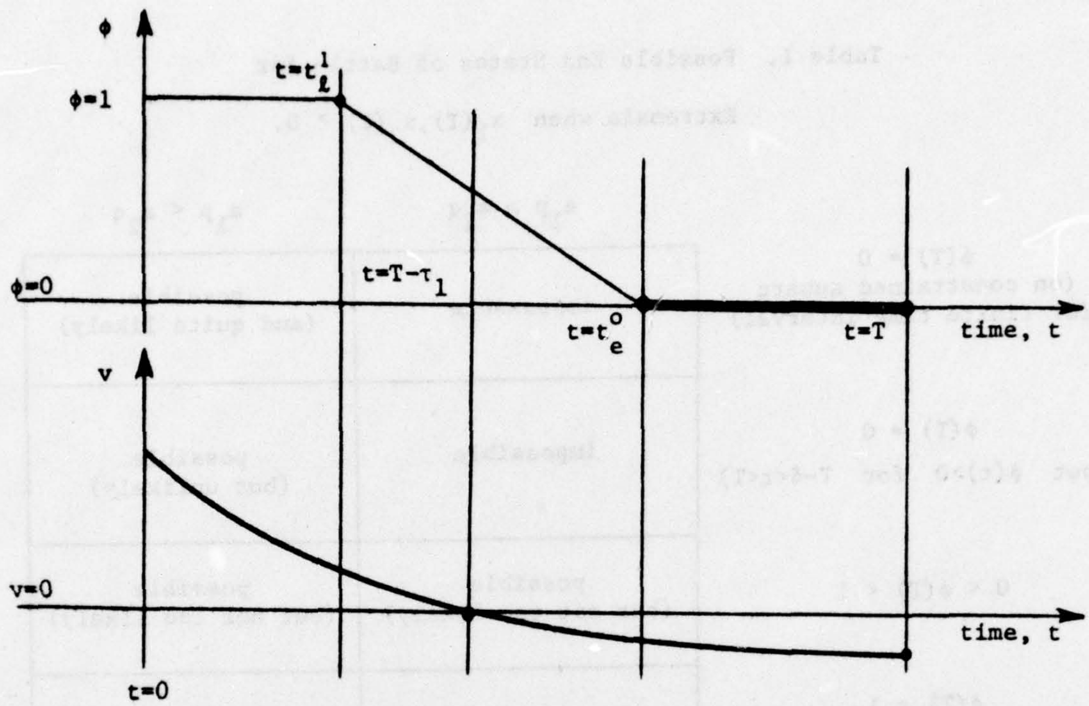

Figure 1. Relationship of t_ℓ^1, $T-\tau_1$, and t_e^o to Values of ϕ and v.

Omitting further details, we reach an important conclusion: for the "inertial" combat (2) one begins to redistribute fire earlier in forward time (anticipating changes in target priority) than in the corresponding "inertialess" case (1). Again, the reader is referred to Figure 1 for motivation of this statement.

The above considerations on tracing extremals backwards from the various terminal states of battle are summarized in Table I. This table shows from which of the end states extremals lead. An entry like "possible (but not too likely)" or "possible (but unlikely)" means that the domain of controllability for extremals* to that end state is a "rather small"

*By the domain of controllability for extremals to an end state we mean the set of points in the initial state space from which extremals lead to the terminal state.

Table I. Possible End States of Battle for Extremals when $x_1(T), x_2(T) > 0$.

	$a_1p \geq a_2q$	$a_1p < a_2q$
$\phi(T) = 0$ (on constrained subarc for finite time interval)	impossible	possible (and quite likely)
$\phi(T) = 0$ (but $\phi(t)>0$ for $T-\delta<t<T$)	impossible	possible (but unlikely)
$0 < \phi(T) < 1$	possible (but not too likely)	possible (but not too likely)
$\phi(T) = 1$ (on constrained subarc for finite time interval)	possible (and quite likely)	impossible
$\phi(T) = 1$ (but $\phi(t)<1$ for $T-\delta<t<T$)	possible (but unlikely)	impossible

subset of the initial state space in contrast to an entry like "possible (and quite likely)" for which the corresponding set of initial conditions is "much larger." It has not been possible to develop explicit formulas for these domains of controllability (as done, for example, in [14], [19]).

6. Synthesis of Extremal Policy when $x_1(T) = 0$ and $x_2(T) > 0$.

There are two cases to consider:

Case (a) on constrained subarc for finite interval of time,

Case (b) $x_1(t) > 0$ for $t < T$.

For Case (a): $x_1(t) = 0$ for $t_1 \le t \le T$ with $t_1 < T$, it is clear that we must have $\phi(t) = 0$ for $t_1 \le t \le T$ so that integration of the adjoint equations (13) through (15) and (63) with the boundary conditions (16) yields that for $t_1 \le t \le T$ we have

$$p_2(t) = -q \cosh\sqrt{a_2 b_2}\,(T-t), \tag{87}$$

$$p_3(t) = q\sqrt{\frac{a_2}{b_2}} \sinh\sqrt{a_2 b_2}\,(T-t), \tag{88}$$

$$v(t) = v(T) + \frac{q}{b_2}(a_1 b_1 - a_2 b_2)\{\cosh\sqrt{a_2 b_2}\,(T-t) - 1\} \quad \text{with} \quad v(T) \ge 0, \tag{89}$$

the requirement that $v(T) \ge 0$ being a consequence of (43). We also have by (40) and (62) that

$$\mu(t) = \mu(T) + \frac{1}{a_1}\int_t^T v(s)ds \quad \text{with} \quad \mu(T) \ge 0. \tag{90}$$

Recalling our nonrestrictive assumption that $a_1 b_1 > a_2 b_2$ and (39), consideration of (89) yields that

$$p_4(t) > 0 \quad \text{for} \quad t_1 \leq t < T \quad \text{with} \quad p_4(t{=}T) = 0,$$

and $$v(t) > 0 \quad \text{for} \quad t_1 \leq t < T \quad \text{with} \quad v(t{=}T) \geq 0.$$

Hence, we see that the necessary conditions of optimality (42) through (44) are always satisfied on the constrained subarc. Thus, there are no restrictions on when such an extremal can occur.

We next show that we must have $v(t_1^-) < 0$. The proof is by contradiction. First, we observe that $x_1(t) > 0$ for $t < t_1$ and $x_1(t{=}t_1) = 0$ imply that

$$u^*(t) = -R_L \quad \text{for} \quad t_1 - \delta < t < t_1 \quad \text{where} \quad \delta > 0. \qquad (91)$$

If we had $v(t_1^-) \geq 0$, then (63) and the fact that $p_3(t) > 0$ for $t < T$ (which follows from (88)) yield that $v(t) > 0$ for $t < t_1$. This yields $p_4(t) > 0$ for $t < t_1$ by (65) and the condition (51) that $p_4(t_1^-) = 0$. However, (91) is impossible if $p_4(t) > 0$.

It should be noted that $v(t_1^-) < 0$ and $v(t_1^+) > 0$ guarantee that $\nu_0 > 0$,* since from (47), (48), and (62) it follows that

$$\nu_0 = \frac{1}{a_1}\{v(t_1^+) - v(t_1^-)\}.$$

It should be recalled that ν_0 is chosen so that $\phi = 0$ when $x_1(t_1) = 0$. The value of ν_0 depends on x_1^o, x_2^o, y_0, ϕ_0, and T_1. The multiplier ν_1 is unrestricted and is chosen so that $v(T) = (a_1p-a_2q) - a_1\nu_1 \geq 0$.

Considering the corner conditions (47), (48), and (51), we have shown that

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*This condition was shown to be necessary for optimality in Section 4.2.

$$p_1(t_1^-) < 0,^* \quad p_2(t_1^-) < 0, \quad p_3(t_1^-) > 0, \quad p_4(t_1^-) = 0, \quad v(t_1^-) < 0. \qquad (92)$$

Further details for $0 \le t < t_1$ in the backwards synthesis are similar to those given for Case (b) in Section 5 and are therefore omitted. Some possibilities for the synthesized extremal control are shown in Table II. Other variations in the form of the synthesized extremal control are possible, and the reader should have no difficulty in identifying them.

For Case (b): $x_1(t) > 0$ for $t < T$, there are three subcases to be considered:

Subcase (b1) $\phi(t=T) = 0$,

Subcase (b2) $0 < \phi(t=T) < 1$,

Subcase (b3) $\phi(t=T) = 1$.

Further analysis yields the results shown in Table III. This shows from which of the end states extremals lead. We will now sketch how these results were obtained.

Subcase (b1): $\phi(t=T) = 0$.

Since $\phi(t=T) = 0$, (16) yields that $p_4(\tau=0) = \nu_4 \ge 0$. It is clear that we must have $\phi(t) > 0$ for $t \in (T-\delta,T)$ for some $\delta > 0$. Hence, by (17) and (20)

$$u^*(t) = -R_L \quad \text{for} \quad t \in (T-\delta_1,T) \subset (T-\delta,T). \qquad (93)$$

We now prove that $p_4(\tau=0) = 0$. The proof is by contradiction. If $p_4(\tau=0) > 0$, then $p_4(t) > 0$ for $t \in (T-\delta_2,T)$ and this contradicts (93)

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*To establish this result one makes the identification $p_1(t) = \frac{\partial W}{\partial x_1(t)}$, where W denotes Bellman's optimal value function. It follows that $p_1(t) < 0$, since addition of more X_1 at t cannot help but reduce Y's optimal return. A justification of this argument is given in [18].

Table II. Some Possibilities for Synthesized Extremal Controls and Collateral Information for Case in which $x_1(T) = 0$ (On Constrained Subarc for Finite Time Interval) and $x_2(T) > 0$ (Must Have $v(T) \geq 0$).

Case (1).

time, t	$p_4(t)$	$v(t)$	$u^*(t)$	$\phi(t)$	$x_1(t)$
$t=0$	>0	>0	R_U	$0\leq\phi<1$	>0
$0<t<t_e^1$	>0	>0	R_U	$0<\phi<1$	>0
$t=t_e^{1-}$	0	>0	R_U	1	>0
$t=t_e^{1+}$	>0	>0	0	1	>0
$t_e^1<t<t_\ell^1$	>0	>0	0	1	>0
$t=t_\ell^{1-}$	0	>0	0	1	>0
$t=t_\ell^{1+}$	0	>0	$-R_L$	1	>0
$t_\ell^1<t<T-\tau_1$	<0	>0	$-R_L$	$0<\phi<1$	>0
$t=T-\tau_1$	<0	0	$-R_L$	$0<\phi<1$	>0
$T-\tau_1<t<t_1$	<0	<0	$-R_L$	$0<\phi<1$	>0
$t=t_1^-$	0	<0	$-R_L$	0	0
$t=t_1^+$	>0	>0	0	0	0
$t_1<t<T$	>0	>0	0	0	0
$t=T$	0	≥0	0	0	0

Case (2).

time, t	$p_4(t)$	$v(t)$	$u^*(t)$	$\phi(t)$	$x_1(t)$
$t=0$	>0	>0	R_U	$0\leq\phi<1$	>0
$0<t<t_s$	>0	>0	R_U	$0<\phi<1$	>0
$t=t_s^-$	0	>0	R_U	$0<\phi\leq1$	>0
$t=t_s^+$	0	>0	$-R_L$	$0<\phi\leq1$	>0
$t_s<t<T-\tau_1$	<0	>0	$-R_L$	$0<\phi<1$	>0
$t=T-\tau_1$	<0	0	$-R_L$	$0<\phi<1$	>0
$T-\tau_1<t<t_1$	<0	<0	$-R_L$	$0<\phi<1$	>0
$t=t_1^-$	0	<0	$-R_L$	0	0
$t=t_1^+$	>0	>0	0	0	0
$t_1<t<T$	>0	>0	0	0	0
$t=T$	0	≥0	0	0	0

Case (3).

time, t	$p_4(t)$	$v(t)$	$u^*(t)$	$\phi(t)$	$x_1(t)$
$t=0$	≤0	>0	$-R_L$	$0<\phi\leq1$	>0
$0<t<T-\tau_1$	<0	>0	$-R_L$	$0<\phi<1$	>0
$t=T-\tau_1$	<0	0	$-R_L$	$0<\phi<1$	>0
$T-\tau_1<t<t_1$	<0	<0	$-R_L$	$0<\phi<1$	>0
$t=t_1^-$	0	<0	$-R_L$	0	0
$t=t_1^+$	>0	>0	0	0	0
$t_1<t<T$	>0	>0	0	0	0
$t=T$	0	≥0	0	0	0

Table III. Possible End States of Battle for Extremals when $x_1(T) = 0$ (But $x_1(t) > 0$ for $t < T$) and $x_2(T) > 0$.

	$a_1 p \geq a_2 q$	$a_1 p < a_2 q$
$\phi(T) = 0$ (on constrained subarc for finite time interval)	impossible	impossible
$\phi(T) = 0$ (but $\phi(t)>0$ for $T-\delta<t<T$)	possible (but very unlikely)	possible (but very unlikely)
$0 < \phi(T) < 1$	possible (but not too likely)	possible (but not too likely)
$\phi(T) = 1$ (on constrained subarc for finite time interval)	possible (and quite likely)	impossible
$\phi(T) = 1$ (but $\phi(t)<1$ for $T-\delta<t<T$)	possible (but unlikely)	impossible

by (17). Furthermore, we must have $v(T) < 0$. Again, the proof is by contradiction. First, we remark that by considering $p_3(t) = \frac{\partial W}{\partial y(t)}$ (see [18] for further discussion) we obtain that $p_3(t) > 0$ for $t < T$. Thus, by (63) we have $\frac{dv}{d\tau}(\tau) > 0$ for $\tau > 0$. If we had $v(T) \geq 0$, then we would have $v(t) > 0$ for $t < T$, and hence $p_4(t) > 0$ for $t \in (T-\delta_2,T)$ via (65) and $p_4(T) = 0$. However, by (17) this contradicts (93).

By (16) we see that $v(T) < 0 \Rightarrow \frac{1}{a_1}(a_1 p - a_2 q) < \nu_1 \leq p$, the latter inequality a consequence of requiring $p_1(T) \leq 0$. It is clear that ν_1 can always be chosen to satisfy the above conditions regardless of whether $a_1 p \geq a_2 q$ or $a_1 p < a_2 q$. Thus, this subcase is always possible for the appropriate initial values of the state variables. Furthermore, we have

$$p_1(T) \leq 0, \quad p_2(T) < 0, \quad p_3(T) = 0, \quad p_4(T) = 0, \quad v(T) < 0,$$

so that the synthesized extremal control function can take any of the forms shown in Table III for $0 \leq t \leq t_1$, the realization of any particular form being dependent upon the state variable initial conditions.

Subcase (b2): $0 < \phi(t=T) < 1$.

Since $0 < \phi(t=T) < 1$, (16) yields that $p_4(\tau=0) = 0$. Moreover, for $0 < \phi(T) \leq 1$ the transversality condition $H(T) = 0$ no longer holds, since when $x_1(T) = 0$ $(=y(T))$, variations in control δu cannot increase T because this would lead to violation of the constraint $x_1 \geq 0$ if the planning horizon were extended to $T + dT$ with $x_1(T) = 0$ and $\phi(T) > 0$. Then, a one-sided version of the usual variational argument [2] yields (after dropping some terms) $dJ = H(T)dT \leq 0$ with $dT \leq 0$, where J denotes the (augmented) return functional. This yields $H(T) \geq 0$ and consequently $p_3(T) \leq 0$. Again using the argument which considers

$p_3(T) = \frac{\partial W}{\partial y(T)}$ [18], it follows that $p_3(T) = 0$ (and also $p_3(t) > 0$ for $t < T$).

Now, $v(\tau=0) = (a_1p-a_2q) - a_1\nu_1$ where $0 \leq \nu_1 \leq p$, the latter inequality being a consequence of requiring $p_1(T) \leq 0$. For $a_1p \geq a_2q$, we can have $v(\tau=0)$ either ≥ 0 or < 0. When $v(\tau=0) \geq 0$, the resulting synthesized extremal control takes a form similar to that shown in Table IV below. When $v(\tau=0) < 0$,* it takes any of the forms shown in Table II for $0 \leq t \leq t_1$. In all cases ν_1 is chosen so that $x_1(T) = 0$. The realization of the synthesized extremal control depends upon the state variable initial conditions.

Subcase (b3): $\phi(t=T) = 1$.

Since $\phi(t=T) = 1$, (16) now yields $p_4(\tau=0) = -\nu_5 \leq 0$. If $\phi(t) < 1$ for $T - \delta \leq t < T$ where $\delta > 0$, then previous arguments (i.e. proof by contradiction) yield that $p_4(\tau=0) = 0$. Next, we show that we must have $v(t=T) \geq 0$. The proof is by contradiction. First, we observe that $\phi(t) < 1$ for $T - \delta \leq t < T$ implies that

$$u^*(t) = R_U \quad \text{for} \quad t \in (T-\delta_1,T) \subset (T-\delta,T). \tag{94}$$

If $v(T) < 0$, then $v(t) < 0$ for $t \in (T-\delta_2,T)$. Considering a Taylor series expansion about $t = T$, we obtain $p_4(t) = (T-t)y(\bar{t})v(\bar{t})$ where $\bar{t} \in (t,T)$, since $p_4(T) = 0$. But then $t \in (T-\delta_2,T) \cap (T-\delta_1,T)$ yields $p_4(t) < 0$ and this contradicts (94) by (17). Since $v(T) = (a_1p-a_2q) - a_1\nu_1 \geq 0$ where $\nu_1 \geq 0$, it follows that this subcase with $\phi(t) < 1$ for $t < T$ is only possible when $a_1p \geq a_2q$.

If we are on a constrained subarc with $\phi(t) = 1$ for $t_e^1 \leq t \leq T$, then (23) and (25) must hold. The former yields $p_4(\tau=0) = 0$, while the

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*This is the only case possible when $a_1p < a_2q$.

latter yields $0 \le v_1 \le \frac{1}{a_1}(a_1p - a_2q)$ so that this subcase with $\phi(t) = 1$ for $t_e^1 \le t \le T$ is only possible when $a_1p \ge a_2q$. The further synthesis of the extremal control now follows previous arguments. The synthesized extremal control is shown in Table IV.

Table IV. Synthesized Extremal Control and Collateral Information for Case in which $x_1(T) = 0$ (But $x_1(t) > 0$ for $t < T$) and $x_2(T) > 0$ with $\phi(t) = 1$ for $t_e^1 \le t \le T$.

time, t	$p_4(t)$	$v(t)$	$u^*(t)$	$\phi(t)$	$x_1(t)$
$t=0$	>0	>0	R_U	$0 \le \phi < 1$	>0
$0<t<t_e^1$	>0	>0	R_U	$0<\phi<1$	>0
$t=t_e^{1-}$	0	>0	0	1	>0
$t=t_e^{1+}$	>0	>0	0	1	>0
$t_e^1<t<T$	>0	>0	0	1	>0
$t=T$	0	≥ 0	0	1	0

Note: This case is only possible when $a_1p \ge a_2q$.

7. Synthesis of Extremal Policy when $x_1(T) > 0$ and $x_2(T) = 0$.

When $x_2(T) = 0$, we must have $x_2(t) > 0$ for $t < T$, since it is nonoptimal (see Section 4.2 above) to be on a constrained subarc with $x_2 = 0$ for a finite interval of time. There are then three cases to be considered:

Case (a) $\phi(t=T) = 0$,

Case (b) $0 < \phi(t=T) < 1$,

Case (c) $\phi(t=T) = 1$.

Further analysis now yields the results shown in Table V. This shows from which of the end states extremals lead. The symmetry of these results (interchange x_1 and x_2) in relation to those shown in Table III for $x_1(T) = 0$ but $x_1(t) > 0$ for $t < T$ should be noted. We will now sketch how these results were obtained.

For Case (a): $\phi(t=T) = 0$, (16) yields that $p_4(\tau=0) = \nu_4 \geq 0$. Observing that $\phi(t) > 0$ for $t \in (T-\delta,T)$ again implies (93), it follows by previous arguments that $p_4(\tau=0) = 0$. Also, (93) implies that $\frac{dp_4}{d\tau}(\tau) < 0$ for $\tau \in (0,\delta_2) \subset (0,\delta_1)$, and hence $v(\tau=0) < 0$ by (65). Observing that $v(\tau=0) = (a_1p-a_2q) + a_2\nu_2$, it follows that this case with $\phi(t) > 0$ for $t < T$ is only possible for $a_1p < a_2q$. The synthesized extremal control takes the form shown in Table II for $0 \leq t \leq t_1$.

If we are on a constrained subarc with $\phi(t) = 0$ for $t_e^o \leq t \leq T$, then (33) and (34) must hold, whence it follows that $p_4(\tau=0) = 0$ and $v(\tau=0) = (a_1p-a_2q) + a_2\nu_2 \leq 0$ or $0 \leq \nu_2 \leq \frac{1}{a_2}(a_2q-a_1p)$. Then $p_2(\tau=0) = -q + \nu_2 < 0$ so that we must have $v(T) < 0$, since for $t_e^o \leq t \leq T$ we have $v(t) = v(T) + (a_1b_1-a_2b_2)\frac{(q-\nu_2)}{b_2}\{\cosh \sqrt{a_2b_2}\,(T-t) - 1\}$ and $v(t) \leq 0$ by (34) and (62). Thus, $0 \leq \nu_2 < \frac{1}{a_2}(a_2q-a_1p)$ so that this case with $\phi(t) = 0$ for $t_e^o \leq t \leq T$ is only possible when $a_1p < a_2q$. The synthesized extremal control is shown in Table VI.

For Case (b): $0 < \phi(t=T) < 1$, (16) yields that $p_4(\tau=0) = 0$. If $a_1p \geq a_2q$, then $v(T) \geq 0$. The requirement that $p_2(T) \leq 0$ yields $0 \leq \nu_2 \leq q$ so that by (14) we have $p_3(t) > 0$ for $t < T$, whence $\frac{dv}{d\tau}(\tau) > 0$ for all $\tau > 0$. Hence, $v(\tau) > 0$ for $\tau > 0$, and the synthesized

Table V. Possible End States of Battle for Extremals when $x_1(T) > 0$ and $x_2(T) = 0$ (But $x_2(t) > 0$ for $t < T$).

	$a_1p \geq a_2q$	$a_1p < a_2q$
$\phi(T) = 0$ (cn constrained subarc for finite time interval)	impossible	possible (and quite likely)
$\phi(T) = 0$ (but $\phi(t)>0$ for $T-\delta<t<T$)	impossible	possible (but unlikely)
$0 < \phi(T) < 1$	possible (but not too likely)	possible (but not too likely)
$\phi(T) = 1$ (on constrained subarc for finite time interval)	impossible	impossible
$\phi(T) = 1$ (but $\phi(t)<1$ for $T-\delta<t<T$)	possible (but very unlikely)	possible (but very unlikely)

Table VI. One Possibility for Synthesized Extremal Control and Collateral Information for Case in which $x_1(T) > 0$ and $x_2(T) = 0$ (but $x_2(t) > 0$ for $t < T$) With $\phi(t) = 0$ for $t_e^o \le t \le T$.

time, t	$p_4(t)$	$v(t)$	$u^*(t)$	$\phi(t)$	$x_2(t)$
$t=0$	>0	>0	R_U	$0 \le \phi < 1$	>0
$0<t<t_e^1$	>0	>0	R_U	$0<\phi<1$	>0
$t=t_e^1$	0	>0	R_U	1	>0
$t=t_e^{1+}$	>0	>0	0	1	>0
$t_e^1<t<t_\ell^1$	>0	>0	0	1	>0
$t=t_\ell^{1-}$	0	>0	0	1	>0
$t=t_\ell^{1+}$	0	>0	$-R_L$	1	>0
$t_\ell^1<t<T-\tau_1$	<0	>0	$-R_L$	$0<\phi<1$	>0
$t=T-\tau_1$	<0	0	$-R_L$	$0<\phi<1$	>0
$T-\tau_1<t<t_e^o$	<0	<0	$-R_L$	$0<\phi<1$	>0
$t=t_e^{o-}$	0	<0	$-R_L$	0	>0
$t=t_e^{o+}$	<0	<0	0	0	>0
$t_e^o<t<T$	<0	<0	0	0	>0
$t=T$	0	<0	0	0	0

Notes: (1) This case is only possible when $a_1 p < a_2 q$.

(2) Variations in the synthesized extremal control analogous to those shown in Table II are possible.

control is as shown in Table IV for $0 \le t \le t_e'$. If $a_1p < a_2q$, then we can have either $v(T) \ge 0$ or $v(T) < 0$. The former case has just been discussed, while the latter case yields a synthesized extremal control like one of those shown in Table II for $0 \le t \le t_1$.

For Case (c): $\phi(t=T) = 1$, (16) yields that $p_4(\tau=0) = -\nu_5 \le 0$. It is clear that we must have $\phi(t) < 1$ for $t \in (T-\delta,T)$, since $x_2(t) > 0$ for $t < T$ and $x_2(T) = 0$. Previous arguments readily yield $p_4(\tau=0) = 0$ and $0 \le \nu_2 \le q$ (since $p_2(T) \le 0$). If $a_1p \ge a_2q$, then $v(T) \ge 0$ and further results are similar to those of the previous case. If $a_1p < a_2q$, then $v(T)$ is either ≥ 0 or < 0. The case in which $v(T) \ge 0$ has just been discussed. If $v(T) < 0$, this leads to $p_4(t) < 0$ for $t \in (T-\delta_2,T)$. By (17) this is impossible, however, since we must have (94) hold, since $\phi(t) < 1$ for $t \in (T-\delta,T)$. Thus, this case is always possible for the appropriate initial values of the state variables, although we must always have $v(T) \ge 0$. The synthesized extremal control is the same as for Case (b) with $v(T) \ge 0$.

8. Determination of the Optimal Fire Distribution Policy.

It remains to discuss how the optimal fire distribution policy may be determined from among the extremal control policies developed in the previous sections. Two ways of proving the optimality of an extremal trajectory are as follows (see [17], [18]):

(a) show that sufficient conditions of optimality are satisfied on the extremal,

(b) by citing the appropriate existence theorem, show that an optimal control exists for the problem at hand; there are two further subcases: (1) if the extremal is unique, then it is optimal or (2) if the extremal is not unique and only a finite number exist,

then the optimal trajectory is determined by considering the finite number of alternatives.

In the first case there are both local and global sufficiency theorems to be considered. Neither, however, is convenient to apply to the problem (2) at hand. The usual control theory sufficient conditions for a local maximum (see pp. 181-184 of [2]) are not satisfied, since the problem (2) is singular (in the sense that $\frac{\partial^2 H}{\partial u^2} \equiv 0$).* Sufficient conditions for a global maximum depend upon the appropriate functions being concave <u>and</u> the planning horizon being <u>fixed</u> in length [4], [9]. The latter condition is not satisfied for (2), since, for example, the battle can end by Y being annihilated at any time before T_1.

Thus, as in earlier papers [14], [18], [19], we are again led back to the enumeration of all extremals in various regions of the state space (i.e. the intersections of the domains of controllability for extremals leading to the various terminal states of battle (see [14], [19])) and comparison of corresponding returns. The author has not been able to develop analytic, closed-form results for the integration of the state equations (2) in the general case (much less for the determination of domains of controllability or computation of extremal returns). Thus, it has not been possible to analytically determine the optimal control from among the candidate extremal controls for the problem at hand** as was done in [14] (see also [19]).

Moreover, the existence of an optimal control readily follows from the control variable u being uniformly bounded (see Corollary 2 on p. 262

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*Recently, conditions sufficient for a local maximum have been developed for the singular control problem. These are, however, essentially impossible to check for the problem at hand (see [18]).

**For given initial and parameter values this may be done numerically by following the steps outlined in [14] (or [15]).

of [8]).* The uniform boundedness of responses is a consequence of u being required to lie in a compact set (i.e. $-\infty < -R_L \leq u \leq R_U < +\infty$. Let us observe that for $-R_L = -\infty$ and $R_U = +\infty$ problem (2) is equivalent to problem (1), the "inertialless" combat problem. In this case there will be jumps in state variable ϕ in problem (2). Additional considerations are now required in the development of necessary conditions of optimality (see [12], [23], [24]). It should be pointed out, moreover, that the existence theorems of Lee and Markus [8] (and others) only apply for admissible trajectories which are absolutely continuous. Hence, they no longer can be invoked to insure the existence of an optimal control when $-R_L = -\infty$ and $R_U = +\infty$.

9. More General Redistribution of Fire Model.

In the model (2) considered above the rate of redistribution of fire by Y firers was assumed to be bounded, i.e. $-R_L \leq \frac{d\phi}{dt} \leq R_U$, and not to be dependent upon the distribution of fire. It is of considerable interest and importance for us to continue our investigation of the dependence of the structure of the optimal fire distribution policy upon model form (see [16]). The above simple model (2) for redistribution of fire is equivalent to

$$\frac{d\phi}{dt} = F_1(u), \tag{95}$$

where (a) $F_1(u=0) = 0$ and (b) $F_1(u)$ is a concave function for $u \in [-R_L, R_U]$ with $F_1'(u=R_L) > 0$.

Let us consider the more general case in which the rate of redistribution of Y-fire is bounded and also dependent upon the distribution of fire. Thus, we consider the model

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*See also [17], [18].

$$\frac{d\phi}{dt} = F(\phi,u), \qquad (96)$$

where $F(\phi,u)$ reflects the ability of Y firers to redistribute their fire over X-target types. We will see below that the functional form of $F(\phi,u)$ (as long as it is not pathological) does not affect the structure of the optimal fire distribution policy, although the time history of the distribution of Y-fire, i.e. $\phi(t)$ for $0 \le t \le T$, does vary, of course, with changes in $F(\phi,u)$.

What are appropriate properties to postulate for $F(\phi,u)$ in order to model the real world? First of all, it must be possible to keep the distribution of fire constant. Thus, we stipulate that if $u = 0$, there is no redistribution of fire. Also, if $u > 0$ (<0), then ϕ increases (decreases) (but the rate of change is bounded). It seems appropriate to postulate that there are "stragglers" in redistributing their fire, as ϕ approaches zero (or one) its rate of change decreases. Finally, we assume that all the Y firers can shift their fire in a finite interval of time.

We therefore make the following <u>assumptions about $F(\phi,u)$</u>:

(A1) for all $\phi \in [0,1]$

$$\begin{cases} -F_L \le F(\phi,u) < 0 & \text{for } -R_L \le u < 0, \\ F(\phi,u) = 0 & \text{for } u = 0, \\ 0 < F(\phi,u) \le F_U & \text{for } 0 < u \le R_U \\ & \text{where } F_L, F_U > 0, \end{cases}$$

(A2) $F(\phi,u)$ is piecewise $C^{(1)}$ in its arguments for all $\phi \in [0,1]$ and $u \in [-R_L, R_U]$,

(A3) for fixed $\phi \in [0,1]$, $F(\phi,u)$ is concave in u for $-R_L \le u \le R_U$ with $\frac{\partial F}{\partial u}(\phi, u{=}R_U) > 0$,

(A4) $F(\phi,u)$ is a strictly $\begin{Bmatrix}\text{increasing}\\ \text{decreasing}\end{Bmatrix}$ function of ϕ for fixed u such that $\begin{Bmatrix}-R_L \le u < 0\\ 0 < u \le R_U\end{Bmatrix}$.

An example of such a function is

$$F(\phi,u) = \begin{cases} u(\varepsilon_1+\phi) & \text{for } -R_L \le u < 0, \\ u(1+\varepsilon_2-\phi) & \text{for } 0 \le u \le R_U, \\ & \text{where } \varepsilon_1, \varepsilon_2 > 0. \end{cases} \tag{97}$$

Then we have, for example, $\phi(t) = -\varepsilon_1 + (\varepsilon_1+\phi_0)e^{-R_L(t-t_0)}$ when $u(s) = -R_L$ for $t_0 \le s \le t$.

We now show that for problem (2) when (96) is used with $F(\phi,u)$ satisfying (A1) through (A4), the structure of the optimal fire distribution policy is the same* as for problem (2) (with $\frac{d\phi}{dt} = u$). The Hamiltonian in this case is given by

$$H(t,x_i,p_i,u) = -p_1\phi a_1 y - p_2(1-\phi)a_2 y - p_3(b_1x_1+b_2x_2) + p_4F(\phi,u)$$

$$- \eta_1(t)u + \eta_2(t)u - \mu(t)\{F(\phi,u)a_1y - \phi a_1(b_1x_1+b_2x_2)\}, \tag{98}$$

so that (15) is now replaced by

$$\frac{dp_4}{dt} = p_1a_1y - p_2a_2y - p_4\frac{\partial F}{\partial \phi} + \mu(t)\left\{\frac{\partial F}{\partial \phi}a_1y - a_1(b_1x_1+b_2x_2)\right\}. \tag{99}$$

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*Of course, $\phi(t)$ and dependent quantities differ in their particular form.

When $x_1, x_2, y > 0$ and $0 < \phi < 1$, the extremal control is again given by (17) because of assumption (A3). Again, it will be shown that there is no singular solution. This is proved as follows. Since $\frac{\partial F}{\partial u} \neq 0$, the condition $\frac{\partial H}{\partial u} = 0$ for a finite interval of time again leads to (18) and $\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0$ on the singular subarc so that (19) again follows. For an arc on which (18) and (19) hold we again have (20) so that $\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) \neq 0$, since it has been shown that $p_3(t) > 0$ for $t < T$, and we have assumed $a_1 b_1 > a_2 b_2$. It follows then that there is no singular solution (i.e. no singular subarc in the solution).

The analysis of constrained subarcs also follows that given above for problem (2). We illustrate this for a constrained subarc on which $x_1(t) = 0$ for $t_e \leq t \leq T$. Again, (36) and (37) hold. Now

$$\frac{\partial H}{\partial u} = \frac{\partial F}{\partial u} (p_4 - \mu a_1 y),$$

so that $\frac{\partial H}{\partial u} = 0$ again yields (38), since $\frac{\partial F}{\partial u} \neq 0$. It is readily shown in a similar fashion that (40) and (41) still hold so that the necessary conditions of optimality on a constrained subarc with $x_1(t) = 0$ for a finite interval of time are again given by (42) through (44). Treatment of other types of constrained subarcs is similar and further discussion is omitted.

Thus, we have shown that the characterization of an optimal fire distribution policy for the more general redistribution of fire model given above is <u>exactly the same</u> as that for problem (2).

10. Discussion.

In this section we discuss what we have learned about the influence of command and control limitations on the structure of optimal fire

distribution policies. The reader should bear in mind, moreover, that the combat model considered in this paper is far too simple to have the results obtained from it be taken literally, but he should interpret these results as indicating general principles or hypotheses to be further investigated by higher resolution analysis techniques such as field experimentation or computer simulation. Nevertheless, it is hoped that our simple model (2) has provided some insights into the optimization of this combat process. Thus, it is the *form* of the *optimal* policy and its functional dependence on model parameters that is of primary interest.

In this paper we have considered a version of the two-on-one fire distribution problem considered elsewhere [14], [18]. In the version (2) considered here there are limits on the rate at which the distribution of fire can change. This might be thought of as reflecting a command and control limitation (e.g. the existence of a time lag between the giving of an order and its implementation). From our above study, we conclude that the structure of the optimal fire distribution policy for problem (2) (a) depends upon the model for the attrition of X-force target types and not upon the nature of the model for redistribution of fire (see Section 9) (as long as this does not change the functional form of enemy target-type attrition); (b) depends upon the following model parameters (see [16] for further discussion): (1) a_1b_1 for $i = 1,2$, (2) a_1p and a_2q, and (3) whether Y wins or loses; and (c) is very similar to that for the inertialess combat case.

To elaborate further about (c), the reader can find results for the inertialess combat problem (1) reported in [14], [18]. When these are compared with those for the problem at hand (2), there are seen to be many

similarities (see below). Both models have the same attrition structure (for X-force target types), although the model (2) considered here incorporates command and control limitations. As long as these do not affect the function form of the attrition process of enemy target types,* the optimal policies are similar (see also Section 9).

We saw that for both "inertialess" and "inertial" combat, we must necessarily have

$$a_1 b_1 \geq a_2 b_2,$$

in order for it to be optimal to drive x_1 to zero (while $x_2 > 0$ and before $t = T$). Furthermore, we also developed a necessary condition involving $v(t) = a_1(-p_1(t)) - a_2(-p_2(t))$ for it to be optimal in (2) to have $\phi(t) = 0$ for 1 for a finite interval of time. Again, the results were similar to those for the "inertialess" combat problem (1).

The sign of the quantity $v(t) = a_1(-p_1(t)) - a_2(-p_2(t))$ reflects the ranking of target priorities: when $v(t) > 0$, X_1 is a higher priority target for Y than is X_2; and the situation is reversed when $v(t) < 0$. A significant result (see Section 5) was that for "inertial" combat an optimal policy for the distribution of Y-fire over enemy target types is characterized by beginning to change the distribution of fire (i.e. shifting of fires) <u>before</u> target priorities (as measured by the sign

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*In [13] Schreiber formulates a Lanchester-type combat model in which the effectiveness of intelligence and command and control systems modifies the form of combatant attrition. These capabilities are incorporated into Schreiber's model through a parameter $\in [0,1]$ which he denotes as "command efficiency." His equations reduce to Lanchester's classical equations for area fire when "command efficiency" is equal to zero for each side and to those for aimed fire when it is equal to one.

of v(t)) change. (It should be recalled that in the "inertialess" version of this problem all fire was concentrated on the target type with the higher priority and instantaneously shifted when this changed.) In other words, due to decreased reaction ability one begins to change the distribution of fire before target priorities change in anticipation of this coming change. We might even generalize this result to say that with slow reactions one's optimal policy involves anticipating enemy actions.

It should finally be pointed out that to the best of the author's knowledge the problem (2) considered in this paper is the first one with a higher order (i.e. greater than first order) SVIC (see [3], [6], [22]) to be considered in the operations research literature.* Moreover, the complete treatment of this problem was made possible by some recent results of the author [21], [22].

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*Examples of problems with an SVIC of order greater than one have appeared, of course, previously in the engineering literature (see [3], [6]).

REFERENCES

[1] O. Bolza, Vorlesungen Über Variationsrechnung, B. G. Teubner, Berlin, 1909 (reprinted by Chelsea, New York, 1963).

[2] A. Bryson and Y. C. Ho, Applied Optimal Control, Blaisdell Publishing Company, Waltham, Massachusetts, 1969.

[3] A. Bryson, W. Denham and S. Dreyfus, "Optimal Programming Problems with Inequality Constraints I: Necessary Conditions for Extremal Solutions," AIAA Journal 1, 2544-2550 (1963).

[4] J. Funk and E. Gilbert, "Some Sufficient Conditions for Optimality in Control Problems with State Space Constraints," SIAM J. Control 8, 498-504 (1970).

[5] L. Graves, "Discontinuous Solutions in the Calculus of Variations," Bull. Amer. Math. Soc. 36, 831-846 (1930).

[6] D. Jacobson, M. Lele and J. Speyer, "New Necessary Conditions of Optimality for Control Problems with State-Variable Inequality Constraints," J. Math. Anal. Appl. 35, 255-284 (1971).

[7] C. Johnson and J. Gibson, "Singular Solutions in Problems of Optimal Control," IEEE Trans. on Automatic Control, Vol. AC-8, 4-15 (1963).

[8] E. Lee and L. Markus, Foundations of Optimal Control Theory, John Wiley & Sons, Inc., New York, 1967.

[9] O. Mangasarian, "Sufficient Conditions for the Optimal Control of Nonlinear Systems," SIAM J. Control 4, 139-152 (1966).

[10] J. McIntyre and B. Paiewonsky, "On Optimal Control with Bounded State Variables," pp. 389-419 in Advances in Control Systems, Vol. 5, C. Leondes, (Ed.), Academic Press, New York, 1967.

[11] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze and E. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York, 1962.

[12] R. Rishel, "An Extended Pontryagin Principle for Control Systems Whose Control Laws Contain Measures," SIAM J. Control 3, 191-205 (1965).

[13] T. Schreiber, "Note on the Combat Value of Intelligence and Command Control Systems," Opns. Res. 12, 507-510 (1964).

[14] J. Taylor, "On the Isbell and Marlow Fire Programming Problem," Naval Res. Log. Quart. 19, 539-556 (1972).

[15] J. Taylor, "Some Differential Games of Tactical Interest," Opns. Res. 22, 304-317 (1974).

[16] J. Taylor, "Lanchester-Type Models of Warfare and Optimal Control," Naval Res. Log. Quart. 21, 79-106 (1974).

[17] J. Taylor, "Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent Attrition-Rate Coefficients," Naval Res. Log. Quart. 21, 683-704 (1974).

[18] J. Taylor, "Target Selection in Lanchester Combat: Linear-Law Attrition Process," Naval Res. Log. Quart. 20, 673-697 (1973).

[19] J. Taylor, "On the Treatment of Force-Level Constraints in Time-Sequential Combat Problems," Naval Res. Log. Quart. 22, 617-650 (1975).

[20] J. Taylor, "Comments on a Multiplier Condition for Problems with State Variable Inequality Constraints," IEEE Trans. on Automatic Control, Vol. AC-17, 743-744 (1972).

[21] J. Taylor, "On Boundary Conditions for Adjoint Variables in Problems with State Variable Inequality Constraints," IEEE Trans. on Automatic Control, Vol. AC-19, 450-452 (1974).

[22] J. Taylor, "On a Multiplier Condition for Problems with p^{th} Order State Variable Inequality Constraints," submitted for publication.

[23] K. Vind, "Control Systems with Jumps in the State Variables," Econometrica 35, 273-277 (1967).

[24] J. Warga, "Variational Problems with Unbounded Controls," SIAM J. Control 3, 424-438 (1966).

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